

# The Rokhlin property and the tracial topological rank \*

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## Abstract

Let  $A$  be a unital separable simple  $C^*$ -algebra with  $\text{TR}(A) \leq 1$  and  $\alpha$  be an automorphism. We show that if  $\alpha$  satisfies the tracially cyclic Rokhlin property then  $\text{TR}(A \rtimes_\alpha \mathbb{Z}) \leq 1$ . We also show that whenever  $A$  has a unique tracial state and  $\alpha^m$  is uniformly outer for each  $m (\neq 0)$  and  $\alpha^r$  is approximately inner for some  $r > 0$ ,  $\alpha$  satisfies the tracial cyclic Rokhlin property. By applying the classification theory of nuclear  $C^*$ -algebras, we use the above result to prove a conjecture of Kishimoto: if  $A$  is a unital simple AT-algebra of real rank zero and  $\alpha \in \text{Aut}(A)$  which is approximately inner and if  $\alpha$  satisfies some Rokhlin property, then the crossed product  $A \rtimes_\alpha \mathbb{Z}$  is again an AT-algebra of real rank zero. As a by-product, we find that one can construct a large class of simple  $C^*$ -algebras with tracial rank one (and zero) from crossed products.

## 1 Introduction

The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by A. Connes ([2]). It was adopted by Herman and Ocneanu ([17]) for UHF-algebras. M. Rørdam ([30]) and A. Kishimoto ([13]) introduced the Rokhlin property to a much more general context of  $C^*$ -algebras

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(see also [9]). A. Kishimoto had been studying automorphisms on UHF-algebras and more generally, on simple  $A\mathbb{T}$ -algebras that satisfy a Rokhlin property ([12], [14]). More recently, N. C. Phillips studied finite group actions which satisfy certain type of Rokhlin property on some simple  $C^*$ -algebras ([26]).

A conjecture of A. Kishimoto can be formulated as follows: Let  $A$  be a unital simple  $A\mathbb{T}$ -algebra of real rank zero and  $\alpha$  be an approximately inner automorphism. Suppose that  $\alpha$  is “sufficiently outer”, then the crossed product of the  $A\mathbb{T}$ -algebra by  $\alpha$ ,  $A \rtimes_\alpha \mathbb{Z}$  is again a unital simple  $A\mathbb{T}$ -algebra. In particular, he studied the case that  $A$  has a unique tracial state.

A. Kishimoto proposed that the appropriate notion of the outerness is the Rokhlin property ([14]). He also introduced the notion of uniformly outer ([13]). In [14], he showed that if  $A$  is a unital simple  $A\mathbb{T}$ -algebra of real rank zero with a unique tracial state and  $\alpha \in \text{Aut}(A)$  is approximately inner, then  $\alpha$  has the Rokhlin property if and only if  $\alpha^m$  is uniformly outer for all  $m \neq 0$ . He also showed that the Rokhlin property, in this situation, is equivalent to say that  $A \rtimes_\alpha \mathbb{Z}$  has real rank zero, and it is equivalent to say that  $A \rtimes_\alpha \mathbb{Z}$  has a unique tracial state. A. Kishimoto showed ([12]) that, if in addition,  $A$  is a UHF-algebra then  $A \rtimes_\alpha \mathbb{Z}$  is in fact a unital simple  $A\mathbb{T}$ -algebra. He also showed in [15] that the conjecture is true for the case that  $A$  is assumed to have a unique tracial state, both  $K_i(A)$  are finitely generated and  $K_1(A) \not\cong \mathbb{Z}$ , and, in addition, that  $\alpha \in \text{Hinn}(A)$ . Among other things, we prove in this paper the Kishimoto conjecture for all cases that  $A$  has a unique tracial state. If the term of “sufficiently outer” is interpreted as “tracial Rokhlin” property, then Kishimoto’s conjecture holds: Let  $A$  be a unital simple  $A\mathbb{T}$ -algebra and  $\alpha$  be an approximate inner automorphism. Suppose that  $\alpha$  has the tracial Rokhlin property, then  $A \rtimes_\alpha \mathbb{T}$  is again a unital simple  $A\mathbb{T}$ -algebra.

We take the advantage of the development in Elliott’s program of the classification of nuclear  $C^*$ -algebras (see [4] and [6], for example). In particular, we use the classification result in [22], where unital separable simple  $C^*$ -algebras satisfying the Universal Coefficient Theorem and with tracial topological rank zero are classified by their  $K$ -theory. Adopting a N. C. Phillips’s observation, we note that if  $A$  is a unital simple  $C^*$ -algebra with  $\text{TR}(A) \leq 1$  and  $\alpha \in \text{Aut}(A)$  satisfies a so-called tracial cyclic Rokhlin property, then  $\text{TR}(A \rtimes_\alpha \mathbb{Z}) \leq 1$  so that the classification result in [22] and [24] can be applied. Using Kishimoto’s techniques, we show that if  $\alpha^r$  is approximately inner (for some integer  $r > 0$ ), the tracial Rokhlin property introduced in ([28]) implies the tracial cyclic Rokhlin property. Using a result in [28], we actually show a more general result (see Theorem 3.5).

The assumption that  $\alpha$  is approximately inner is to insure that the crossed products remain finite (see the introduction of [14]). We relax this restriction slightly by only requiring that  $\alpha^r$  is approximately inner for some integer  $r > 0$ . It turns out that in a number of cases, while there are automorphisms which

are not approximate inner, all (outer) automorphisms  $\alpha$  have this property, i.e., for some integer  $r > 0$ ,  $\alpha^r$  are approximately inner (see Theorem [H.2](#)). We show that our results also cover many cases in which  $A$  may have arbitrary tracial space (see Corollary [H.4](#)).

It is shown by G. Gong ([\[8\]](#)) that a unital simple AH-algebra with very slow dimension growth has tracial topological rank one or zero. Moreover, Elliott, Gong and Li ([\[7\]](#)) show that the class of unital simple AH-algebras with very slow dimension growth can be classified by their  $K$ -theoretical data. An improvement of this classification has been made so that unital simple nuclear  $C^*$ -algebras with tracial topological rank no more than one which satisfy the Universal Coefficient Theorem can also be classified by their  $K$ -theoretical data ([\[24\]](#)). However, until now, all interesting examples of unital simple nuclear  $C^*$ -algebras that have tracial topological rank one are those AH-algebras with very slow dimension growth (and those of similar inductive limit construction). Theorem [E.7](#) also provides ways to construct unital simple  $C^*$ -algebras with tracial topological rank one by crossed products (see Corollary [H.4](#) and Example [H.5](#)). It also creates the opportunity to apply the classification results in [\[24\]](#).

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## 2 The Rokhlin properties

The following conventions will be used in this paper. Let  $A$  be a unital  $C^*$ -algebra

- (i) We denote by  $\text{Aut}(A)$  the set of all automorphisms on  $A$  and by  $T(A)$  the tracial state space of  $A$ .
- (ii) Two projections  $p, q \in A$  are said to be equivalent if they are Murray-von Neumann equivalent. That is, there exists a partial isometry  $w \in A$  such that  $w^*w = p$  and  $ww^* = q$ . Then we write  $p \sim q$ .
- (iii) Let  $\mathcal{F}$  and  $\mathcal{S}$  be subsets of  $A$  and  $\varepsilon > 0$ . We write  $x \in_\varepsilon \mathcal{S}$  if there exists  $y \in \mathcal{S}$  such that  $\|x - y\| < \varepsilon$ , and write  $\mathcal{F} \subset_\varepsilon \mathcal{S}$  if  $x \in_\varepsilon \mathcal{S}$  for all  $x \in \mathcal{F}$ .
- (iv) Let  $a$  and  $b$  be two positive elements in  $A$ . We write  $[a] \leq [b]$  if there exists an element  $x \in A$  such that  $a = x^*x$  and  $xx^* \in bAb$ . If  $ab = ba = 0$ , then we write  $[a + b] = [a] + [b]$ . Let  $p$  be a projection and  $b$  non-zero

positive element in  $A$ . Note that  $[p] \leq [b]$  implies that  $p$  is Murray-von Neumann equivalent to a projection in the hereditary  $C^*$ -algebra  $\overline{bAb}$ .

- (v) We denote by  $\mathcal{I}^{(0)}$  the class of all finite dimensional  $C^*$ -algebras, and by  $\mathcal{I}^{(k)}$  the class of all  $C^*$ -algebras with the form  $pM_n(C(X))p$ , where  $X$  is a finite CW complex with dimension  $k$  and  $p \in M_n(C(X))$  is a projection.

We recall the definition of tracial topological rank of  $C^*$ -algebras.

**ID0** **Definition 2.1.** [Lntro<sup>0</sup>] Let  $A$  be a unital simple  $C^*$ -algebra and  $k \in \mathbb{N}$ . Then  $A$  is said to have *tracial topological rank no more than  $k$*  if and only if for any finite set  $\mathcal{F} \subset A$ , and  $\varepsilon > 0$  and any non-zero positive element  $a \in A$ , there exists a  $C^*$ -subalgebra  $B \subset A$  with  $B \in \mathcal{I}^{(k)}$  and  $\text{id}_B = p$  such that

- (1)  $\|[x, p]\| < \varepsilon$  for all  $x \in \mathcal{F}$ ,
- (2)  $pxp \in_{\varepsilon} B$  for all  $x \in \mathcal{F}$ ,
- (3)  $[1 - p] \leq [a]$ .

We write  $\text{TR}(A) \leq k$ .

**IR0** **Remark 2.2.** [Lntro<sup>0</sup>] Corollary 6.15] Let  $A$  be a simple unital  $C^*$ -algebra with stable rank one which satisfies the Fundamental Comparison Property. Then  $\text{TR}(A) \leq k$  if and only if for any finite set  $\mathcal{F} \subset A$ ,  $\varepsilon > 0$ , and any non-zero positive element  $a \in A$ , there exists a  $C^*$ -subalgebra  $B \subset A$  with  $B \in \mathcal{I}^{(k)}$  and  $\text{id}_B = p$  such that

- (1)  $\|[x, p]\| < \varepsilon$  for all  $x \in \mathcal{F}$ ,
- (2)  $pxp \in_{\varepsilon} B$  for all  $x \in \mathcal{F}$ ,
- (3)  $\tau(1 - p) < \varepsilon$  for all  $\tau \in \text{T}(A)$ .

Recall that  $A$  is said to have the *Fundamental Comparison Property* if  $p, q \in A$  are two projections with  $\tau(p) < \tau(q)$  for all  $\tau \in \text{T}(A)$ , then  $p$  is equivalent to a subprojection of  $q$ .

The following is defined in [Pp<sup>0</sup>], Definition 2.1].

**ID1** **Definition 2.3.** Let  $A$  be a simple unital  $C^*$ -algebra and let  $\alpha \in \text{Aut}(A)$ . We say  $\alpha$  has the *tracial Rokhlin property* if for every finite set  $F \subset A$ , every  $\varepsilon > 0$ , every  $n \in \mathbb{N}$ , and every nonzero positive element  $x \in A$ , there are mutually orthogonal projections  $e_0, e_1, \dots, e_n \in A$  such that:

- (1)  $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n-1$ .
- (2)  $\|e_j a - ae_j\| < \varepsilon$  for  $0 \leq j \leq n$  and all  $a \in F$ .
- (3) With  $e = \sum_{j=0}^n e_j$ ,  $[1-e] \leq [x]$ .

We define a slightly stronger version of the tracial Rokhlin property similar to the approximately Rokhlin property in [\[T2, Definition 4.2\]](#).

**ID2** **Definition 2.4.** Let  $A$  be a simple unital  $C^*$ -algebra and let  $\alpha \in \text{Aut}(A)$ . We say  $\alpha$  has the *tracial cyclic Rokhlin property* if for every finite set  $F \subset A$ , every  $\varepsilon > 0$ , every  $n \in \mathbb{N}$ , and every nonzero positive element  $x \in A$ , there are mutually orthogonal projections  $e_0, e_1, \dots, e_n \in A$  such that

- (1)  $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n$ , where  $e_{n+1} = e_0$ .
- (2)  $\|e_j a - ae_j\| < \varepsilon$  for  $0 \leq j \leq n$  and all  $a \in F$ .
- (3) With  $e = \sum_{j=0}^n e_j$ ,  $[1-e] \leq [x]$ .

**IR1** **Remark 2.5.** (i) The only difference between the tracial Rokhlin property and the tracial cyclic Rokhlin property is that in condition (1) we require that  $\|\alpha(e_n) - e_0\| < \varepsilon$ .

(ii) If  $A$  has real rank zero, stable rank one and has weakly unperforated  $K_0(A)$  (or if  $A$  has SP-property, stable rank one, and the Fundamental Comparison Property), then condition (3) in both Rokhlin property can be replaced by the following condition (3)' using the standard argument:

(3)' With  $e = \sum_{j=0}^n e_j$ , we have  $\tau(1-e) < \varepsilon$  for all  $\tau \in T(A)$ .

(iii) If  $A$  is a simple unital  $C^*$ -algebra with real rank zero, stable rank one, and has weakly unperforated  $K_0(A)$ , the Rokhlin property in the sense of Kishimoto ([\[T2\]](#)) implies the tracial Rokhlin property ([\[28\]](#)).

Recall that a  $C^*$ -algebra  $A$  is said to have *SP-property* if any non-zero hereditary  $C^*$ -subalgebra of  $A$  has a non-zero projection.

Obviously the tracial cyclic Rokhlin property implies the tracial Rokhlin property. The converse is also true in many cases. We will discuss it in the next section.

Before stating the characterization of the tracial Rokhlin property we cite the following notion introduced by Kishimoto ([\[13\]](#)).

**ID3** **Definition 2.6.** Let  $A$  be a unital  $C^*$ -algebra and  $\alpha \in \text{Aut}(A)$ . We say  $\alpha$  is uniformly outer if for any  $a \in A$ , any projection  $p \in A$ , and any  $\varepsilon > 0$ , there are finite number of projections  $p_1, \dots, p_n$  in  $A$  such that  $\sum_i p_i = p$  and  $\|p_i a \alpha(p_i)\| < \varepsilon$  for  $i = 1, \dots, n$ .

The following result is the tracial Rokhlin version of Kishimoto's result in the case of simple unital AT-algebras with a unique trace [IT2, Theorem 2.1].

**[IT1]** **Theorem 2.7.** [PP28] Let  $A$  be a simple unital  $C^*$ -algebra with  $\text{TR}(A) = 0$ , and suppose that  $A$  has a unique tracial state. Then the following conditions are equivalent:

- (1)  $\alpha$  has the tracial Rokhlin property.
- (2)  $\alpha^m$  is not weakly inner in the GNS representation  $\pi_\tau$  for any  $m \neq 0$ .
- (3)  $A \rtimes_\alpha \mathbb{Z}$  has real rank zero.
- (4)  $A \rtimes_\alpha \mathbb{Z}$  has a unique trace.

Note that the uniformly outerness implies that  $\alpha$  is not weakly inner in the GNS representation  $\pi_\tau$  by an  $\alpha$ -invariant tracial state  $\tau$  on  $A$  by [Ks1, Lemma 4.4].

**[IR2]** **Remark 2.8.** When  $A$  is a simple unital  $C^*$ -algebra with tracial topological rank zero, if  $\alpha \in \text{Aut}(A)$  has the tracial Rokhlin property, it is proved in [PP28] that the crossed product  $A \rtimes_\alpha \mathbb{Z}$  has real rank zero, stable rank one, and the order on projections over  $A \rtimes_\alpha \mathbb{Z}$  is determined by traces. But it is not known that the crossed product  $A \rtimes_\alpha \mathbb{Z}$  has tracial topological rank zero. However, if  $\alpha$  has the tracial cyclic Rokhlin property, then we have the following result based on an observation of N. C. Phillips ([P26]).

**[IT2]** **Theorem 2.9.** Let  $A$  be a simple unital  $C^*$ -algebra with  $\text{TR}(A) \leq 1$ . Suppose that  $\alpha \in \text{Aut}(A)$  has the tracial cyclic Rokhlin property. Then  $\text{TR}(A \rtimes_\alpha \mathbb{Z}) \leq 1$ .

In particular, if  $A$  has  $\text{TR}(A) = 0$ , then  $\text{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$ .

*Proof.* We first note that, by [IT],  $A \rtimes_\alpha \mathbb{Z}$  is a simple  $C^*$ -algebra.

Let  $\varepsilon > 0$ ,  $n \in \mathbb{N}$ , and  $F \subset A \rtimes_\alpha \mathbb{Z}$  be a finite set. To simplify notation, without loss of generality, we may assume that

$$F = \{a_i\}_{i=1}^m \cup \{u\},$$

where  $a_i \in A$  and  $\|a_i\| \leq 1$  ( $i = 1, 2, \dots, m$ ) and  $u$  is a unitary which implements  $\alpha$ . Fix  $b \in (A \rtimes_\alpha \mathbb{Z})_+ \setminus \{0\}$ .

Since  $A$  has SP-property and  $\alpha$  is outer,  $A \rtimes_\alpha \mathbb{Z}$  also has SP-property [IT0, Theorem 4]. In particular, there is a non-zero projection  $r \in \overline{b(A \rtimes_\alpha \mathbb{Z})b}$ . Let  $r_0 \in A$  be a nonzero projection. Since  $A \rtimes_\alpha \mathbb{Z}$  is simple, by 1.8 of [3], it is easy to find a non-zero projection  $r' \in r_0 Ar_0$  such that  $r'$  is equivalent to a subprojection of  $r$  (see, for example, [IT0, Theorem 4]). Hence there

are projections  $r_1, r_2 \in A$  such that  $r_1 r_2 = 0$  and  $r_1 + r_2$  is equivalent to a subprojection of  $r$  (see, for example, 3.5.7 of [23]).

Since  $\alpha$  has the tracial cyclic Rokhlin property, for any  $\delta > 0$  with  $\delta < \frac{\varepsilon}{5}$  there exist projections  $e_1, e_2$  such that

- (1)  $\|\alpha(e_i) - e_{i+1}\| < \delta$  for  $1 \leq i \leq 2$  ( $e_3 = e_1$ )
- (2)  $\|[e_i, a_k]\| < \delta$  for  $1 \leq k \leq m$ .
- (3)  $[1 - e_1 - e_2] \leq [r_1]$ .

Set  $p = e_1 + e_2$ . From (1) above, one estimates that

$$\begin{aligned} \|up - pu\| &= \left\| \sum_{i=1}^2 ue_i - \sum_{i=1}^2 e_{i+1}u \right\| \\ &= \sum_{i=1}^2 \|ue_i - e_{i+1}u\| < 2\delta. \end{aligned}$$

Hence, together with (2) above, we obtain

- (4)  $\|[p, a]\| < 2\delta$  for all  $a \in F$ .

There is a unitary  $v \in A \rtimes_\alpha \mathbb{Z}$  such that  $\|v - 1\| < \delta$  and  $vu^*e_iuv^* = e_{i+1}$  for  $1 \leq i \leq 2$ . Set  $w = vu^*$ , and consider the  $C^*$ -algebra  $D$  generated by  $e_1Ae_1$  and  $e_2we_1$ . Then  $D$  is isomorphic to  $e_1Ae_1 \otimes M_2(\mathbb{C})$ . Note that  $pw = e_1w + e_2w = we_2 + we_1 = wp$ . Moreover,  $pwp \in D$ . Since  $\|pup - pwp\| < \delta$ , one has that  $pup \in_{\delta} D$ . By (2) again, we have

$$\|pa_jp - (e_1a_je_1 + e_2a_je_2)\| < 2\delta \quad j = 1, 2, \dots, m.$$

It follows that  $pFp \subset_{2\delta} D$ .

Since  $A$  is a simple  $C^*$ -algebra with SP-property, there exists a nonzero projection  $r_3 \in e_1Ae_1$  such that  $r_3$  is equivalent to a subprojection of  $r_2$ . Since  $\text{TR}(e_1Ae_1) \leq 1$  ([20, Theorem 5.3]). So there exists a  $C^*$ -subalgebra  $B \in \mathcal{I}^{(k)}$  ( $k = 1$  or  $k = 0$ ) and projection  $e = 1_B$  such that

- (5)  $\|[pap, e]\| < \delta < \varepsilon$  for all  $a \in F$ ,
- (6)  $G \subset_{\delta} B$  and
- (7)  $[p - e] \leq [r_3]$ .

From (3), (4), (5), and (7) above we estimate that

- (8) For any  $f \in F$

$$\begin{aligned} \|ef - fe\| &= \|e(pf - fp) + efp - pfe + (pf - fp)e\| \\ &\leq \|pf - fp\| + \|epfp - pfpe\| + \|pf - fp\| \\ &< 5\delta < \varepsilon \quad \text{and} \end{aligned}$$

(9)

$$\begin{aligned}[1 - e] &= [1 - p + p - e], \\ &= [1 - p] + [p - e] \\ &\leq [r_1] + [r_3] \leq [r_1] + [r_2] \leq [r] \leq [b].\end{aligned}$$

From (6) and  $pFp \subset_{2\delta} D$ , we have

(10)  $pFp \subset_{4\delta} D$ .

Hence from the estimates (8), (9), (10) we conclude that  $A \rtimes_{\alpha} \mathbb{Z}$  has tracial topological less than or equal to 1.

In the case of  $\text{TR}(A) = 0$   $B$  can be chosen to be finite dimensional. Hence, in that case,  $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$ .  $\square$

### 3 Approximately inner automorphisms

IIL1

**Lemma 3.1.** Let  $A$  be a unital separable  $C^*$ -algebra and  $\alpha : A \rightarrow A$  be an approximate inner automorphism. Suppose that  $\{p_j\}$  is a central sequence of projections. Then there exists a central sequence of partial isometries  $\{w_j\}$  such that  $w_j^* w_j = p_j$  and  $w_j w_j^* = \alpha(p_j)$ ,  $j = 1, 2, \dots$

*Proof.* Fix a finite subset  $\mathcal{F} \subset A$  which is in the unit ball of  $A$ . Let  $\varepsilon > 0$ . Choose a unitary  $v \in U(A)$  such that

$$\|\alpha(a) - v^* a v\| < \varepsilon/8 \text{ for all } a \in \mathcal{F}.$$

Since  $\alpha$  is an automorphism,  $\alpha(p_j)$  is also a central sequence of projections. Choose a sufficiently large  $j$  so that

$$\|p_j a - a p_j\| < \varepsilon/8 \text{ for all } a \in \mathcal{F} \quad \text{and} \quad \|\alpha(p_j)v - v\alpha(p_j)\| < \varepsilon/8.$$

Since  $\alpha$  is approximately inner, we obtain another unitary  $z \in U(A)$  such that

$$\|z^* p_j z - \alpha(p_j)\| < \varepsilon/8 \quad \text{and} \quad \|z^* a z - \alpha(a)\| < \varepsilon/8 \text{ for all } a \in \mathcal{F}.$$

It follows that

$$\|(vz^*)p_j(zv^*) - \alpha(p_j)\| \leq \|vz^* p_j z v^* - v\alpha(p_j)v^*\| + \|v\alpha(p_j)v^* - \alpha(p_j)\| < \varepsilon/4.$$

Let  $x_j = vz^* p_j$ . Then  $x_j^* x_j = p_j$  and

$$\|x_j x_j^* - \alpha(p_j)\| < \varepsilon/4.$$

From the above we also have

$$\|vz^* az v^* - a\| < \varepsilon/4 \quad \text{and} \quad \|vz^* a - avz^*\| < \varepsilon/4 \text{ for all } a \in \mathcal{F}.$$

On the other hand, for any  $a \in \mathcal{F}$ ,

$$\|x_j a - ax_j\| \leq \|vz^* p_j a - vz^* a p_j\| + \|vz^* a p_j - avz^* p_j\| < \varepsilon/8 + \varepsilon/4 = 3\varepsilon/8.$$

There is a unitary  $u \in U(A)$  such that  $\|u - 1\| < \varepsilon/4$  such that

$$u(x_j x_j^*) u^* = \alpha(p_j).$$

Define  $w_j = ux_j$ . Then  $w_j w_j^* = p_j$  and  $w_j w_j^* = \alpha(p_j)$ . Moreover we have that

$$\|w_j a - aw_j\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Since  $\mathcal{F}$  is arbitrary, the lemma follows.  $\square$

**IIL2** **Lemma 3.2.** Let  $A$  be a unital separable  $C^*$ -algebra and  $\alpha \in \text{Aut}(A)$  for which  $\alpha^r$  is approximately inner for some integer  $r \geq 1$ . Let  $m \in \mathbb{N}$ ,  $m_0 \geq m$  be the smallest integer such that  $m_0 = 0 \bmod r$  and  $l = m + (r-1)(m_0+1)$ .

Suppose that  $\{e_i^{(n)}\}$ ,  $i = 0, 1, \dots, l$ ,  $n = 1, 2, \dots$ , are  $l+1$  sequences of projections in  $A$  satisfying the following:

$$\|\alpha(e_i^{(n)}) - e_{i+1}^{(n)}\| < \delta_n, \quad \lim_{n \rightarrow \infty} \delta_n = 0,$$

$$e_i^{(n)} e_j^{(n)} = 0, \text{ if } i \neq j, \quad e_i^{(n)} \sim e_j^{(n)} \text{ in } A,$$

and for each  $i$ ,  $\{e_i^{(n)}\}$  is a central sequence.

Then for each  $i = 0, 1, 2, \dots, m$ , there is a central sequence of partial isometries  $\{w_i^{(n)}\}$  such that

$$(w_i^{(n)})^* w_i^{(n)} = p_i^{(n)} \quad \text{and} \quad w_i^{(n)} (w_i^{(n)})^* = p_{i+1}^{(n)}, \quad i = 0, 1, \dots, m-1,$$

where  $p_i^{(n)} = \sum_{j=0}^{r-1} e_{i+j(m_0+1)}^{(n)}$ . Moreover, for each  $i$ ,

$$\lim_{n \rightarrow \infty} \|\alpha(p_i^{(n)}) - p_{i+1}^{(n)}\| = 0.$$

*Proof.* Since  $\alpha^r$  is approximately inner, by applying Lemma [IIL1](#), for each  $i, j = 0, 1, \dots, l$ , one obtains central sequences of partial isometries  $\{z(i, j, n)\}$  such that

$$z(i, j, n)^* z(i, j, n) = e_i^{(n)} \quad \text{and} \quad z(i, j, n) z(i, j, n)^* = \alpha^{rj}(e_i^{(n)}).$$

Note that

$$\|\alpha^{rj}(e_i^{(n)}) - e_{i+rj}^{(n)}\| < rj\delta_n.$$

There is a unitary  $u(i, j, n) \in U(A)$ , for each  $i$  and  $j$ , such that

$$\|u(i, j, n) - 1\| < 2(rl)\delta_n \quad \text{and} \quad u(i, j, n)^* \alpha^{rj}(e_i^{(n)}) u(i, j, n) = e_{i+rj}^{(n)}$$

Since  $\lim_{n \rightarrow \infty} \delta_n = 0$ , for each  $i$  and  $j$ ,  $\{u(i, j, n)\}$  is central. Therefore, to simplify notation, we may assume that

$$z(i, j, n)^* z(i, j, n) = e_i^{(n)} \quad \text{and} \quad z(i, j, n) z(i, j, n)^* = e_{i+rj}^{(n)}.$$

Define

$$p_i^{(n)} = \sum_{j=0}^{r-1} e_{i+j(m_0+1)}^{(n)}.$$

Then for each  $i$ , one checks easily that there are central sequences of partial isometries  $\{w(i, n)\}$  such that

$$w(i, n)^* w(i, n) = p_i^{(n)} \quad \text{and} \quad w(i, n) w(i, n)^* = p_{i+1}^{(n)}, \quad i = 0, 1, \dots, m.$$

For example, (with  $m_0 = kr$ ), one defines

$$w(0, n) = z(1, k, n)^* + z(1+(m_0+1), k, n)^* + \cdots + z(1+(r-2)(m_0+1), k, n)^* + z(0, (r-1)k+1, n).$$

Then (with  $m_0 = kr$ )

$$\begin{aligned} & w(0, n) w(0, n)^* \\ &= z(1, k, n)^* z(1, k, n) + z(1+(m_0+1), k, n)^* z(1+(m_0+1), k, n) + \cdots \\ &+ z(1+(r-2)(m_0+1), k, n)^* z(1+(r-2)(m_0+1), k, n) \\ &+ z(0, (r-1)k+1, n) z(0, (r-1)k+1, n)^* \\ &= e_1^{(n)} + e_{1+(m_0+1)}^{(n)} + \cdots + e_{1+(r-2)(m_0+1)}^{(n)} + e_{1+(r-1)(m_0+1)}^{(n)} \\ &= p_1^{(n)} \end{aligned}$$

(note that  $e_{((r-1)k+1)r}^{(n)} = e_{(r-1)kr+r}^{(n)} = e_{(r-1)m_0+r}^{(n)} = e_{1+(r-1)(m_0+1)}^{(n)}$  and

$$\begin{aligned} & w(0, n)^* w(0, n) \\ &= z(1, k, n) z(1, k, n)^* + z(1+(m_0+1), k, n) z(1+(m_0+1), k, n)^* + \cdots \\ &+ z(1+(r-2)(m_0+1), k, n) z(1+(r-2)(m_0+1), k, n)^* \\ &+ z(0, (r-1)k+1, n)^* z(0, (r-1)k+1, n) \\ &= e_{1+kr}^{(n)} + e_{1+(m_0+1)+kr}^{(n)} + \cdots + e_{1+(r-2)(m_0+1)+kr}^{(n)} + e_0^{(n)} \\ &= e_{m_0+1}^{(n)} + e_{2(m_0+1)}^{(n)} + \cdots + e_{(r-1)(m_0+1)}^{(n)} + e_0^{(n)} \\ &= p_0^{(n)}, \quad \text{and} \end{aligned}$$

Since, for each  $i$  and  $j$ ,  $\{z(i, j, n)\}$  is central, so is  $\{w(i, n)\}$ .

From the construction we know that for each  $i$

$$\begin{aligned}\|\alpha(p_i^{(n)}) - p_{i+1}^{(n)}\| &\leq \sum_{j=0}^{r-1} \|\alpha(e_{i+j(m_0+1)}) - e_{i+1+j(m_0+1)}\| \\ &\leq r\delta_n \rightarrow 0 \quad (n \rightarrow \infty)\end{aligned}$$

□

Let  $\{E_{i,j}\}$  be a system of matrix units and  $\mathcal{K}$  be the compact operators on  $\ell^2(\mathbb{Z})$  where we identify  $E_{i,i}$  with the one-dimensional projection onto the functions supported by  $\{i\} \subset \mathbb{Z}$ . Let  $S$  be the canonical shift operator on  $\ell^2(\mathbb{Z})$ . Define an automorphism  $\sigma$  of  $\mathcal{K}$  by  $\sigma(x) = SxS^*$  for all  $x \in \mathcal{K}$ . Then  $\sigma(E_{i,j}) = E_{i+1,j+1}$ . For any  $N \in \mathbb{N}$  let  $P_N = \sum_{i=0}^{N-1} E_{i,i}$ .

LK **Lemma 3.3.** (Kishimoto, 2.1 of [12]) For any  $\eta > 0$  and  $n \in \mathbb{N}$  there exist  $N \in \mathbb{N}$  and projections  $e_0, e_1, \dots, e_{n-1}$  in  $\mathcal{K}$  such that

$$\begin{aligned}\sum_{i=0}^{n-1} e_i &\leq P_N \\ \|\sigma(e_i) - e_{i+1}\| &< \eta, \quad i = 0, \dots, n-1, \quad e_n = e_0 \\ \frac{n \dim e_0}{N} &> 1 - \eta.\end{aligned}$$

IIT1 **Theorem 3.4.** Let  $A$  be a unital separable simple  $C^*$ -algebra with  $\text{TR}(A) \leq 1$  and  $\alpha^r \in \text{Aut}(A)$  be an approximately inner automorphism for some integer  $r \geq 1$ . Suppose that  $\alpha$  has the tracial Rokhlin property then  $\alpha$  has the tracial cyclic Rokhlin property.

*Proof.* Let  $\varepsilon > 0$ . Let  $\varepsilon/2 > \eta > 0$  and  $m \in \mathbb{N}$  be given. Choose  $N$  which satisfies the conclusion of Lemma 3.3 (with this  $\eta$  and  $n = m$ ). Identify  $P_N \mathcal{K} P_N$  with  $M_N$ . Let  $\mathcal{G} = \{E_{i+1,i} : i = 0, 1, \dots, N-1\}$  be a set of generators of  $M_N$ . Let  $e_0, e_1, \dots, e_{m-1}$  be as in the conclusion of Lemma 3.3.

For any  $\varepsilon > 0$ , there is  $\delta > 0$  depends only on  $N$  such that, if

$$\|ag - ga\| < \delta$$

for  $g \in \mathcal{G}$ , then

$$\|ae_i - e_i a\| < \varepsilon/2, \quad i = 0, 1, \dots, n.$$

We assume that  $\delta < \eta$ . Fix a finite subset  $\mathcal{F}_0 \subset A$ .

Choose  $m_0 \in \mathbb{N}$  such that  $m_0 \geq m$  is the smallest integer with  $m_0 = 0 \pmod{r}$ . Let  $L = N + (r-1)(m_0+1)$ .

Since  $\alpha$  has the tracial Rokhlin property, there exists a sequence of projections  $\{e_i^{(k)} : i = 0, 1, \dots, L\}$  satisfying the following:

$$\|\alpha(e_i^{(k)}) - e_{i+1}^{(k)}\| < \frac{\delta}{(2^k)4N}, \quad e_i^{(k)}e_j^{(k)} = 0, \text{ if } i \neq j,$$

$$\lim_{k \rightarrow \infty} \|e_i^{(k)}a - ae_i^{(k)}\| = 0 \text{ for all } a \in A, i = 0, 1, \dots, L \quad \text{and}$$

$$\tau\left(1 - \sum_{i=0}^{L-1} e_i^{(k)}\right) < \eta \text{ for all } \tau \in T(A), k = 1, 2, \dots$$

By applying Lemma [B.2](#), we obtain a central sequence  $\{w_i^{(k)}\}$  in  $A$  such that

$$(w_i^{(k)})^* w_i^{(k)} = P_0^{(k)} \quad \text{and}$$

$$w_i^{(k)}(w_i^{(k)})^* = P_i^{(k)}, \quad k = 0, 1, \dots, i = 0, 1, \dots, N,$$

$$P_i^{(k)} P_j^{(k)} = 0 \quad i \neq j,$$

$$\|\alpha(P_i^{(k)}) - P_{i+1}^{(k)}\| < \frac{\delta}{4L}, \quad k = 0, 1, \dots, i = 0, 1, \dots, N-1,$$

$$\tau\left(1 - \sum_{i=0}^{N-1} P_i^{(k)}\right) < \eta, \quad \text{for all } \tau \in T(A)$$

where  $P_i^{(k)} = \sum_{j=0}^{r-1} e_{i+(m_0+1)j}^{(k)}$  for  $i = 0, 1, \dots, N$ .

It follows that  $\{\alpha^l(w^{(k)})\}_{l=0}^{N-1}$  are all central sequences. As the same argument in Lemma [B.2](#) there is a unitary  $u_k \in U(A)$  with  $\|u_k - 1\| < \delta/2N$  such that  $\text{ad } u_k \circ \alpha(P_i^{(k)}) = P_{i+1}^{(k)}$ ,  $i = 0, 1, \dots, N-1$ . Put  $\beta_k = \text{ad } u_k \circ \alpha$ , and  $w^{(k)} = w_0^{(k)}$ . Choose a large  $k$ , such that

$$\|\beta_k^l(w^{(k)})a - a\beta_k^l(w^{(k)})\| < \delta \text{ for all } a \in \mathcal{F}_0,$$

$l = 0, 1, \dots, N$ .

Now let  $C_1$  and  $C_2$  be the  $C^*$ -algebras generated by  $w^{(k)}, \beta_k^1(w^{(k)}), \dots, \beta_k^{N-1}(w^{(k)})$  and by  $w^{(k)}, \beta_k^1(w^{(k)}), \dots, \beta_k^N(w^{(k)})$ , respectively. Note that  $C_1 \cong M_N$ ,  $C_2 \cong M_{N+1}$ . Define a homomorphism  $\Phi : C_1 \rightarrow \mathcal{K}$  by

$$\Phi(\beta_k^i(w^{(k)})) = E_{i+1,i}, \quad i = 0, 1, \dots, N-1$$

(see Lemma [B.3](#)). Then one has  $\sigma \circ \Phi|_{C_1} = \Phi \circ \beta_k|_{C_1}$  and  $\Phi(C_1) = P_N \mathcal{K} P_N$ . Now we apply Lemma [B.3](#) to obtain mutually orthogonal projections  $e_0, e_1, \dots, e_{m-1}$  in  $M_N$  such that

$$\|\sigma(e_i) - e_{i-1}\| < \eta \quad \text{and} \quad \frac{m \dim e_0}{N} > 1 - \eta.$$

Let  $p_i = \Phi^{-1}(e_i)$ ,  $i = 0, 1, \dots, m-1$ . One estimates that

$$\tau\left(\sum_{i=0}^{N-1} P_i^{(k)} - \sum_{i=0}^{m-1} p_i\right) < 1 - \sum_{i=0}^{m-1} \frac{\dim(e_0)}{N} = 1 - \frac{m \dim(e_0)}{N} < \eta < \frac{\varepsilon}{2}$$

for all  $\tau \in T(A)$ . So one has mutually orthogonal projections  $p_0, p_1, p_2, \dots, p_{m-1}$  such that

$$\|\beta_k(p_i) - p_{i+1}\| < \frac{\varepsilon}{2}, \quad i = 0, 1, 2, \dots, m-1, \quad p_m = p_0.$$

By the choice of  $\delta$ , one also has

$$\|ap_i - p_i a\| < \varepsilon, \quad i = 0, 1, \dots, m-1, \quad \text{for all } a \in \mathcal{F}_0 \quad \text{and}$$

$$\tau\left(1 - \sum_{i=0}^{m-1} p_i\right) < \tau\left(1 - \sum_{i=0}^{N-1} P_i^{(k)}\right) + \frac{\varepsilon}{2} < \eta + \frac{\varepsilon}{2} < \varepsilon,$$

for all  $\tau \in T(A)$ . Since

$$\|\beta_k - \alpha\| < \delta/2 < \varepsilon/2,$$

one finally has

$$\|\alpha(p_i) - p_{i+1}\| < \varepsilon, \quad i = 0, 1, \dots, m-1, \quad p_m = p_0.$$

In other words,  $\alpha$  has the tracial cyclic Rokhlin property.  $\square$

IIT2

**Theorem 3.5.** Let  $A$  be a unital separable simple  $C^*$ -algebra with  $\mathrm{TR}(A) = 0$  which has a unique tracial state and satisfies the Universal Coefficient Theorem. Suppose that  $\alpha^r \in \mathrm{Aut}(A)$  is approximately inner for some integer  $r \geq 1$  and that  $\alpha^m$  is uniformly outer for any integer  $m \neq 0$ . Then  $A \rtimes_\alpha \mathbb{Z}$  is a simple AH-algebra with no dimension growth with real rank zero.

*Proof.* Note that since  $\alpha$  is outer,  $A \rtimes_\alpha \mathbb{Z}$  is simple by [IT]. From Theorem IIT1  $\alpha$  has the tracial Rokhlin property. Since  $\alpha^r \in \mathrm{Aut}(A)$  is approximately inner for some integer  $r \geq 1$ , this implies that  $\alpha$  has the tracial cyclic Rokhlin property by Theorem B.4. So from Theorem IIT2  $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$ . Using the classification theorem of [Imsri] we conclude that  $A \rtimes_\alpha \mathbb{Z}$  is a simple AH-algebra with no dimension growth with real rank zero.  $\square$

The following shows that the Kishimoto's conjecture that we mentioned in the introduction is true at least for the case that the simple AT-algebra has a unique tracial state. In Corollary IITC2, we show that if one agrees that the "sufficiently outer" means the automorphism has tracially Rokhlin property then we do not need to assume that  $A$  has a unique tracial state.

**IIC1** **Corollary 3.6.** Let  $A$  be a unital simple AT-algebra with a unique trace and real rank zero, and let  $\alpha \in \text{Aut}(A)$  such that  $\alpha$  is approximately inner. If  $\alpha^m$  is uniformly outer for any integer  $m \neq 0$ , or  $\alpha$  has tracial Rokhlin property, then  $A \rtimes_\alpha \mathbb{Z}$  is a unital simple AT-algebra of real rank zero.

*Proof.* From the following Pimsner-Voiculescu exact sequence [27],

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\text{id}-\alpha_*^{-1}} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\text{id}-\alpha_*^{-1}} & K_1(A). \end{array}$$

We see that  $K_0(A \rtimes_\alpha \mathbb{Z})$  and  $K_1(A \rtimes_\alpha \mathbb{Z})$  are torsion free. From Theorem B.7, Theorem B.4, and Theorem B.9 we know that  $\text{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$  and  $A \rtimes_\alpha \mathbb{Z}$  satisfies the UCT. Therefore  $K_0(A \rtimes_\alpha \mathbb{Z})$  is a weakly unperforated Riesz group. It follows from [5] that there is a unital simple AT-algebra  $B$  with real rank zero which has the same ordered scaled  $K$ -theory of  $A \rtimes_\alpha \mathbb{Z}$ . It follows from Theorem 5.1 of [22] that  $A \cong B$ .  $\square$

**IIC2** **Corollary 3.7.** Let  $A$  be a unital simple AT-algebra (with real rank zero) and  $\alpha \in \text{Aut}(A)$ . Suppose that  $\alpha$  is approximately inner and  $\alpha$  has the tracial Rokhlin property. Then  $A \rtimes_\alpha \mathbb{Z}$  is a unital simple AT-algebra (with real rank zero.)

*Proof.* It follows from Theorem B.4 that  $\alpha$  actually has the tracial cyclic Rokhlin property. Then, by Theorem B.9,  $\text{TR}(A \rtimes_\alpha \mathbb{Z}) \leq 1$ . As in the proof of IIC1,  $A \rtimes_\alpha \mathbb{Z}$  has torsion free  $K$ -theory. We then apply the classification theorem in [22] (for real rank zero case) or apply [24] (for real rank one case) to conclude that  $A \rtimes_\alpha \mathbb{Z}$  is a unital simple AT-algebra (and with real rank zero).  $\square$

**Remark 3.8.** Kishimoto in [12], [14] and [15] proved that if  $A$  is a simple unital AT-algebra of real rank zero with a unique trace, and  $\alpha \in \text{Aut}(A)$  is an approximately inner with the Rokhlin property, then  $A \rtimes_\alpha \mathbb{Z}$  is also a simple unital AT-algebra under the assumption that both  $K_0(A)$  and  $K_1(A)$  are finitely generated with  $K_1(A) \neq \mathbb{Z}$  and  $\alpha \in \text{HInn}(A)$ . Corollary B.6 shows that the extra conditions of  $K_*(A)$  and  $\alpha \in \text{HInn}(A)$  are not necessary. Corollary B.7 shows that Kishimoto's conjecture holds in general (without assuming that  $A$  has the unique tracial state) if the "sufficiently outer" is replaced by the tracial Rokhlin property. One should note that the tracial Rokhlin property is weaker than the Rokhlin property used in Kishimoto's work. (See Remark B.5(iii).) Moreover, tracially cyclic Rokhlin property is related to "

approximate Rokhlin” property in 4.2 of [12] which is also weaker than the Rokhlin property used in Kishimoto’s work. If one allows the “sufficiently outer” replaced by tracially cyclic Rokhlin property, then  $A \rtimes_{\alpha} \mathbb{Z}$  is always a unital simple AT-algebra without even assuming that  $\alpha^r$  is approximately inner but assuming  $A \rtimes_{\alpha} \mathbb{Z}$  has torsion free  $K$ -theory.

**IIIIf** **Theorem 3.9.** *Let  $A$  be a unital separable simple  $C^*$ -algebra with  $\text{TR}(A) = 0$  or  $\text{TR}(A) = 1$  and  $\alpha \in \text{Aut}(A)$  such that  $\alpha^r$  is approximately inner for some integer  $r > 0$ . Suppose that  $\alpha$  has tracial Rokhlin property. Then  $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$ , or  $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 1$ . Furthermore, if, in addition,  $A$  satisfies the Universal Coefficient Theorem, then  $A \rtimes_{\alpha} \mathbb{Z}$  is a simple AH-algebra with no dimension growth.*

*Proof.* The first part follows from Theorem B.4 and Theorem B.9. For the last part, by [24],  $A$  is a simple AH-algebra with no dimension growth. By the first part,  $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) \leq 1$ , it follows from [24] again that  $A \rtimes_{\alpha} \mathbb{Z}$  is also a simple AH-algebra with no dimension growth.  $\square$

**IIIF** **Remark 3.10.** In Theorem B.4, we assume that  $\text{TR}(A) \leq 1$ . In fact, we only need to assume that  $A$  has the property (SP) and has the Fundamental Comparison Property. Suppose that  $A$  is a unital separable simple  $C^*$ -algebra with  $\text{TR}(A) = 0$  and with a unique tracial state. Suppose also that  $A \rtimes_{\alpha} \mathbb{Z}$  has a unique tracial state (unique ergodic). Then by applying Theorem B.4, Theorem B.9 and Theorem B.11,  $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$ . On the other hand, in Corollary B.6 and Corollary B.7, if we assume only that  $\alpha^r$  is approximate inner for some integer  $r > 1$  and  $\alpha$  has the tracial Rokhlin property, then  $A \rtimes_{\alpha} \mathbb{Z}$  may not be an AT-algebra. This is because  $A \rtimes_{\alpha} \mathbb{Z}$  may have torsion. However, it is a unital AH-algebra with no dimension growth by Theorem B.5. But, in Corollary B.6 and Corollary B.7, if we assume that  $\alpha^r$  is approximate inner for some integer  $r$  and  $A \rtimes_{\alpha} \mathbb{Z}$  has torsion free  $K$ -theory, then conclusion of both Corollary B.6 and Corollary B.7 hold. To allow torsion, related to the Kishimoto’s conjecture, we proved (in Theorem B.9) the following: If  $A$  is a unital simple AH-algebra with no dimension growth (with real rank zero) and  $\alpha \in \text{Aut}(A)$  has the tracial Rokhlin property and  $\alpha^r$  is approximate inner for some integer  $r > 0$ , then  $A \rtimes_{\alpha} \mathbb{Z}$  is again a unital simple AH-algebra with no dimension growth (and with real rank zero).

## 4 Examples

Let  $G$  and  $F$  be abelian groups. Recall that  $Pext(G, F)$  is the subgroup of those extensions

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$$

so that each finitely generated subgroup of  $G$  lifts. If  $A$  is a separable  $C^*$ -algebra which satisfies the Universal Coefficient Theorem, then, for any  $\sigma$ -unital  $C^*$ -algebra  $B$ ,  $KL(A, B) = KK(A, B)/Pext(K_*(A), K_{*-1}(B))$ .

**IVLKK**

**Lemma 4.1.** Let  $A$  be a separable amenable  $C^*$ -algebra satisfying the UCT. Suppose that  $\alpha \in \text{Aut}(A)$  such that  $(\alpha)_{*i} = \text{id}_{K_i(A)}$ ,  $i = 0, 1$ . Suppose that  $\text{ext}_{\mathbb{Z}}(K_{i-1}(A), K_i(A))/Pext(K_{i-1}(A), K_i(A))$  is finite. Then there are integers  $r > 0$  and  $k > 0$  such that

$$[\alpha^{r+k}] = [\alpha^k] \text{ in } KL(A, A).$$

*Proof.* Consider  $[\alpha^m] - [\alpha]$ , for  $m = 1, 2, \dots$ . Since  $(\alpha)_{*i} = \text{id}_{K_i(A)}$ ,  $i = 0, 1$ , by the Universal Coefficient Theorem ([31]), one computes that

$$[\alpha^m] - [\alpha] \in \text{ext}_{\mathbb{Z}}(K_{*+1}(A), K_*(A)).$$

Since  $\text{ext}_{\mathbb{Z}}(K_{i-1}(A), K_i(A))/Pext(K_{i-1}(A), K_i(A))$  is finite, there are positive integers  $r$  and  $k$  such that

$$([\alpha^{r+k}] - [\alpha]) = ([\alpha^k] - [\alpha]) \text{ in } KL(A, A).$$

It follows that

$$[\alpha^{r+k}] = [\alpha^k] \text{ in } KL(A, A)$$

□

**IVTainn**

**Theorem 4.2.** Let  $A$  be a unital separable simple  $C^*$ -algebra with  $\text{TR}(A) = 0$  satisfying the UCT. Suppose that  $\alpha \in \text{Aut}(A)$ . In any of the following cases,  $\alpha^r$  is approximately inner for some integer  $r > 0$ . Consequently, if  $\alpha^m$  is uniformly outer for all  $m \in \mathbb{Z} \setminus \{0\}$  (or  $\alpha$  has tracial Rokhlin property),  $\alpha$  has tracial cyclic Rokhlin property and  $\text{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$ . In particular,  $A \rtimes_{\alpha} \mathbb{Z}$  is a simple AH-algebra with no dimension growth and real rank zero.

- (1)  $K_0(A) = D$ , where  $D$  is a countable dense subgroup of  $\mathbb{R}$  and  $K_1(A) = \mathbb{Z}$ , or  $K_1(A) = \{0\}$ ;
- (2)  $K_0(A) = D$ , where  $D$  is a finitely generated countable dense subgroup of  $\mathbb{R}$  and  $K_1(A) = \mathbb{Z}$  or  $K_1(A)$  is finite;
- (3)  $K_0(A) = D \oplus G$ , with

$$K_0(A)_+ = \{(r, x) \mid r \in D, r > 0, x \in G\} \cup \{(0, 0)\},$$

and  $D$  is a dense subgroup of  $\mathbb{R}$  such that for any non-zero element  $d \in D$  and any integer  $n \geq 1$ , there is  $e \in D$  such that  $me = d$  for some  $m \geq n$ , where  $G = \mathbb{Z}$  or  $G$  is finite and  $K_1(A) = \mathbb{Z}$ , or  $K_1(A) = \{0\}$ ;

- (4)  $K_0(A) = \mathbb{Q} \oplus G$ , where  $G = \mathbb{Z}$  or  $G$  is finite and  $K_1(A) = \mathbb{Z}$ , or  $K_1(A)$  is a finite group.

*Proof.* In all cases, it suffices to show that  $\alpha^r$  is approximately inner for some integer  $r \geq 1$ .

For (1), it is clear that  $\alpha_{*0} = \text{id}_{K_0(A)}$ . If  $K_1(A) = \mathbb{Z}$ , since  $\alpha_{*1}$  is an isomorphism,  $\alpha_{*1}(1) = \pm 1$ . Therefore  $\alpha_{*i}^2 = \text{id}_{K_i(A)}$ . Since  $K_i(A)$  are torsion free,  $[\alpha^2] = [\text{id}_A]$  in  $KL(A, A)$ . It follows from Theorem 2.4 of [21] that  $\alpha^2$  is approximately inner.

For (2), as in (1),  $\alpha_{*0} = \text{id}_{K_0(A)}$ . Also if  $K_1(A) = \mathbb{Z}$ , then  $\alpha_{*1}^2 = \text{id}_{K_1(A)}$ . If  $K_1(A)$  is finite, since  $\alpha_{*1}$  is an isomorphism, there exists  $r_1 \geq 1$  such that  $\alpha_{*1}^{r_1} = \text{id}_{K_1(A)}$ . Let  $\beta = \alpha^{2r_1}$ . Then  $\beta_{*i} = \text{id}_{K_i(A)}$ ,  $i = 0, 1$ . However, in this case,

$$\text{ext}_{\mathbb{Z}}(D, K_1(A)) = \{0\} \quad \text{and} \quad \text{ext}_{\mathbb{Z}}(K_1(A), K_0(A)) \text{ is finite.}$$

It follows from Lemma H.1 that  $[\beta^{m+k}] = [\beta^k]$  in  $KL(A, A)$  for some integer  $m, k \geq 1$ . By Theorem 2.3 of [21], there exists a sequence of unitaries such that

$$\lim_{n \rightarrow \infty} \text{ad } u_n \circ \beta^k(a) = \beta^{m+k}(a) \text{ for all } a \in A.$$

Since  $\beta^k$  is an automorphism, it follows that  $\beta^m$  is approximately inner, or  $\alpha^{m(2r_1)}$  is approximately inner.

For (3), as above, one has that  $\alpha_{*1}^2 = \text{id}_{K_1(A)}$ . The assumption on  $D$  implies that there is no nonzero homomorphism from  $D$  to  $\mathbb{Z}$  or a finite group. One then checks that there is an integer  $r_1 \geq 1$  such that  $\alpha_{*0}^{r_1} = \text{id}_{K_0(A)}$ . Put  $\beta = \alpha^{2r_1}$ . Then  $\beta_{*i} = \text{id}_{K_i(A)}$ ,  $i = 0, 1$ . To see that  $\beta^m$  is approximately inner, we note that  $\text{ext}_{\mathbb{Z}}(D, K_1(A)) = P\text{ext}(D, K_1(A))$  since  $D$  is torsion free. One then computes that

$$\text{ext}_{\mathbb{Z}}(K_0(A), K_1(A))/P\text{ext}(K_0(A), K_1(A)) = \text{ext}_{\mathbb{Z}}(G, K_1(A))/P\text{ext}(K_0(A), K_1(A))$$

which is finite, and  $\text{ext}_{\mathbb{Z}}(K_1(A), K_0(A)) = \{0\}$ . Thus one can apply the same argument as in the case (2) by applying Lemma H.1.

For (4), as in the case (2) and (3), there is  $r_1 \geq 1$  such that  $\alpha_{*i}^{r_1} = \text{id}_{K_i(A)}$ ,  $i = 0, 1$ . Moreover, since  $\mathbb{Q}$  is divisible,

$$\text{ext}_{\mathbb{Z}}(K_1(A), \mathbb{Q}) = \{0\}.$$

Because  $K_1(A) = \mathbb{Z}$ , or  $K_1(A)$  is finite,

$$\text{ext}_{\mathbb{Z}}(K_1(A), K_0(A)) = \text{ext}_{\mathbb{Z}}(K_1(A), G) \text{ is also finite.}$$

Since  $\mathbb{Q}$  is torsion free,  $\text{ext}_{\mathbb{Z}}(\mathbb{Q}, K_1(A)) = P\text{ext}(\mathbb{Q}, K_1(A))$ . One then computes that

$$\text{ext}_{\mathbb{Z}}(K_0(A), K_1(A))/P\text{ext}(K_0(A), K_1(A)) = \text{ext}_{\mathbb{Z}}(G, K_1(A))/P\text{ext}(K_0(A), K_1(A))$$

which is finite. Thus the argument in the case (3) applies.  $\square$

**IIIP1**

**Proposition 4.3.** Let  $A$  be a simple unital  $C^*$ -algebra with  $\text{TR}(A) = 0$ , and  $B$  be a simple unital  $C^*$ -algebra with  $\text{TR}(B) \leq 1$ . Suppose that  $\alpha \in \text{Aut}(A)$  has the tracial cyclic Rokhlin property. Then for any  $\beta \in \text{Aut}(B)$   $\alpha \otimes \beta \in \text{Aut}(A \otimes_{\min} B)$  has the tracial cyclic Rokhlin property.

*Proof.* It follows from  $\text{HLX}$   $\text{TR}(A \otimes_{\min} B) \leq 1$ . Hence  $A \otimes_{\min} B$  has  $\text{SP}_{\text{Ltr}_0}$  property, stable rank one and the Fundamental Comparison Property ( $\text{FCP}$ , Proposition 6.2 and Theorems 6.9 and 6.11]).

Let  $F \subset A \otimes_{\min} B$  be a finite set,  $n \in \mathbb{N}$ , and  $\varepsilon > 0$ . Without loss of generality, we may assume that there exist a finite set  $F_A \subset A$  and  $F_B \subset B$  such that  $F = F_A \otimes F_B$ .

Since  $\alpha$  has the tracial cyclic Rokhlin property, there exist mutually orthogonal projections  $e_0, e_1, \dots, e_n \in A$  such that

- (1)  $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n$ , where  $e_{n+1} = e_0$ .
- (2)  $\|e_j a - a e_j\| < \varepsilon$  for  $0 \leq j \leq n-1$  and all  $a \in F_A$ .
- (3)  $\tau(1 - \sum_{j=0}^n e_j) < \varepsilon$  for all tracial states  $\tau$  on  $A$ .

(See Remark  $\text{L}^{\text{R1}}_2$  (ii).)

Set  $f_i = e_i \otimes 1_B$  for  $0 \leq i \leq n$ . Then  $f_i$  are mutually orthogonal projections in  $A \otimes_{\min} B$  such that

- (1)  $\|(\alpha \otimes \beta)(f_j) - f_{j+1}\| < \varepsilon$  for  $0 \leq j \leq n$ , where  $f_{n+1} = f_0$ .
- (2)  $\|f_j a - a f_j\| < \varepsilon$  for  $0 \leq j \leq n-1$  and all  $a \in F$ .
- (3)  $\tau(1 - \sum_{j=0}^n f_j) < \varepsilon$  for all tracial states  $\tau$  on  $A \otimes_{\min} B$ .

This means that  $\alpha \otimes \beta$  has the tracial cyclic Rokhlin property by Remark  $\text{L}^{\text{R1}}_2$  (ii).  $\square$

**IVCtr=1**

**Corollary 4.4.** Let  $A$  be a separable simple amenable unital  $C^*$ -algebra with  $\text{TR}(A) = 0$  which satisfies the UCT, and  $B$  be a simple amenable unital  $C^*$ -algebra with  $\text{TR}(B) \leq 1$ . Suppose also that  $A$  has a unique tracial state and  $\alpha \in \text{Aut}(A)$  such that  $\alpha^m$  is uniformly outer for all  $m \neq 0$  and  $\alpha^r$  is approximately inner for some integer  $r \geq 1$ . Then for any  $\beta \in \text{Aut}(B)$ ,  $\alpha \otimes \beta$  has the tracial cyclic Rokhlin property and  $\text{TR}(D) \leq 1$ , where  $D = (A \otimes B) \rtimes_{\alpha \otimes \beta} \mathbb{Z}$ .

An unexpected consequence of the above corollary is that it provides a new way to construct unital simple  $C^*$ -algebras with tracial topological rank one. All previous examples are inductive limit construction (see  $\text{[8]}$ ). Since there is basically no restriction on  $B$  and  $\beta$ , a great number of those simple

$C^*$ -algebras  $D$  with  $\text{TR}(D) = 1$  can be obtained from Corollary 4.4. Since  $\text{TR}(B) = 1$ , one certainly expects that most such  $D$  has  $\text{TR}(D) = 1$  but not  $\text{TR}(D) = 0$ . To convince the reader that it is likely the case, we compute the tracial rank in a very special case below. From its construction, it should be clear how other example can be constructed.

Denote  $\text{Aff}(A)$  the space of all affine continuous functions on  $\text{T}(A)$ . Given a projection  $p \in M_n(A)$  for some integer  $n \geq 1$  we define  $\rho_A(p)(\tau) = (\tau \otimes \text{Tr})(p)$  for all  $\tau \in \text{T}(A)$ , where  $\text{Tr}$  is the standard trace on  $M_n(\mathbb{C})$ . Then  $\rho_A(p) \in \text{Aff}(\text{T}(A))$ .

**IV.E Example 4.5.** Let  $A$  be a unital UHF-algebra with  $K_0(A) = \mathbb{Q}$  and  $\alpha$  be in  $\text{Aut}(A)$  so that  $\alpha^m$  is uniformly outer for all  $m \neq 0$  (or  $\alpha$  is uniquely ergodic). Then  $\alpha$  has the tracial cyclic Rokhlin property by [T2, Lemma 4.3] and Theorem 3.5. Let  $B$  be a unital simple  $A\mathbb{T}$ -algebra for which  $K_0(B) = \mathbb{Q}$  and  $K_1(B) = \mathbb{Z} \oplus \mathbb{Z}$  and  $\text{Aff}(\text{T}(B)) = C_{\mathbb{R}}([0, 1])$ . Existence of such simple  $A\mathbb{T}$ -algebra was given by [32]. It follows from section 9 of [32] that there is  $\beta \in \text{Aut}(B)$  such that  $\beta_{*1}(x, y) = (-x, y)$  for  $(x, y) \in \mathbb{Z} \oplus \mathbb{Z}$ ,  $\beta_{*0} = \text{id}_{K_0(B)}$  and  $\tau \circ \beta(b) = \tau(b)$  for all  $b \in B$  and  $\tau \in \text{T}(B)$ . It should be noted that  $\tau \circ \alpha(a) = \tau(a)$  for all  $a \in \text{T}(A)$ . Put  $\gamma = \alpha \otimes \beta$  and  $C = A \otimes B$  and  $D = C \rtimes_{\gamma} \mathbb{Z}$ .

By the Künneth formula one computes that  $C$  is a unital simple ( $A\mathbb{T}$ -algebra) with  $K_0(C) = \mathbb{Q}$  and  $K_1(C) = \mathbb{Q} \oplus \mathbb{Q}$ . One computes that  $\gamma_{*0} = \text{id}_{K_0(C)}$  and  $\gamma_{*1}((x, y)) = (-x, y)$  for  $(x, y) \in \mathbb{Q} \oplus \mathbb{Q}$ .

It follows from Proposition 4.3 that  $\gamma$  has the tracial cyclic Rokhlin property. Moreover  $\text{TR}(D) \leq 1$ . To check that  $\text{TR}(D) = 1$ , we first compute that, by Pimsner-Voiculescu's exact sequence and by the divisibility of  $\mathbb{Q}$ , we have  $K_0(D) = \mathbb{Q} \oplus \mathbb{Q}$  and  $K_1(D) = \mathbb{Q} \oplus \mathbb{Q}$ . Consider tracial states with the form  $t \otimes \tau$ , where  $t \in \text{T}(A)$  and  $\tau \in \text{T}(B)$ . Note all these tracial states are  $\gamma$  invariant. Thus they give tracial states on  $D$ . Note that  $\text{T}(A)$  is a single point. Thus we may identify  $\text{T}(B)$  with  $\text{T}(A) \otimes \text{T}(B)$ . Hence  $\text{Aff}(\text{T}(A) \otimes \text{T}(B)) = C_{\mathbb{R}}([0, 1])$ . Let  $e_1 = \rho_D((1, 0))$  and  $e_2 = \rho_D((0, 1))$ . Then

$$\rho_D(K_0(D)) = \{xe_1 + ye_2 : x, y \in \mathbb{Q}\}.$$

We view  $\text{T}(B) \subset \text{T}(D)$ . Thus one has a surjective affine homomorphism  $\Lambda : \text{Aff}(\text{T}(D)) \rightarrow \text{Aff}(\text{T}(B))$ . It is easy to see that  $\Lambda \circ \rho_D(K_0(D))$  being rank two can not be dense in  $C_{\mathbb{R}}([0, 1])$ . It follows from [T, Theorem 6.9] that  $D$  has real rank other than zero (actually one). It follows from Theorem 7.1(c) of [20] that  $\text{TR}(D) = 1$ .

If one insists to get non-zero torsion in  $K$ -theory, one may start, for example, with  $K_0(A) = \mathbb{Q}$  and  $K_1(A) = \mathbb{Z}/p\mathbb{Z}$ .

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